Math 53: Multivariable Calculus **Sections 102, 103**

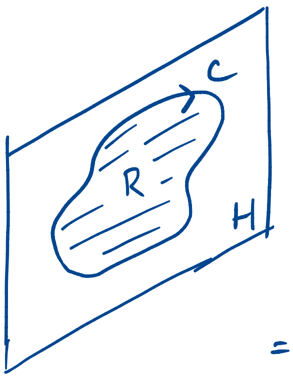
Worksheet for 2020-04-29

Problems

$2x + 0y - z + 4 = 0$

Problem 1. Throughout this problem, let H denote the plane $z = 2x + 4$.

(a) Let $\mathbf{F} = \langle 3yz, xz, xy - yz\rangle$. Show that if C is any oriented simple closed curve contained in the plane H, then $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$, regardless of C.



$$\int_C \vec{F}\cdot d\vec{r} = \iint_R (\nabla\times\vec{F})\cdot d\vec{S}$$

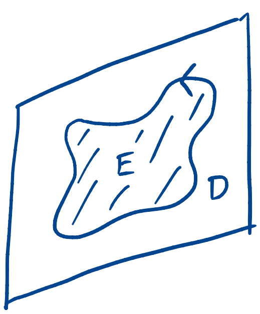
$$\nabla\times\vec{F} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3yz & xz & xy-yz \end{bmatrix} = \langle -z, 2y, -2z\rangle$$

$$= \iint_R \langle -z, 2y, -2z\rangle\cdot\hat{n}\, dS$$

$$= \pm \iint_R \langle -z, 2y, -2z\rangle\cdot\frac{\langle 2,0,-1\rangle}{\sqrt{5}}\, dS = \iint_R 0\, dS = 0.$$

depending on orientation of C.

(b) Let $\mathbf{G} = \langle x^2y - y, 0, y^3/6\rangle$. If we let D to be any simple closed curve contained in the plane H which is oriented *counterclockwise* when viewed from above, find the maximum possible value of the integral $\int_D \mathbf{G}\cdot d\mathbf{r}$.



$$\nabla\times\vec{G} = \left\langle \frac{y^2}{2}, 0, 1-x^2\right\rangle$$

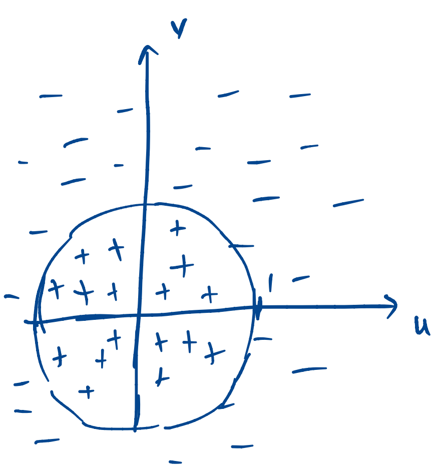
$z = 2x+4$

$\vec{r}(u,v) = \langle u, v, 2u+4\rangle$

$$\int_D \vec{G}\cdot d\vec{r} = \iint_E \left\langle \frac{y^2}{2}, 0, 1-x^2\right\rangle\cdot d\vec{S}$$

$$\vec{r}_u\times\vec{r}_v = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} = \langle -2, 0, 1\rangle$$

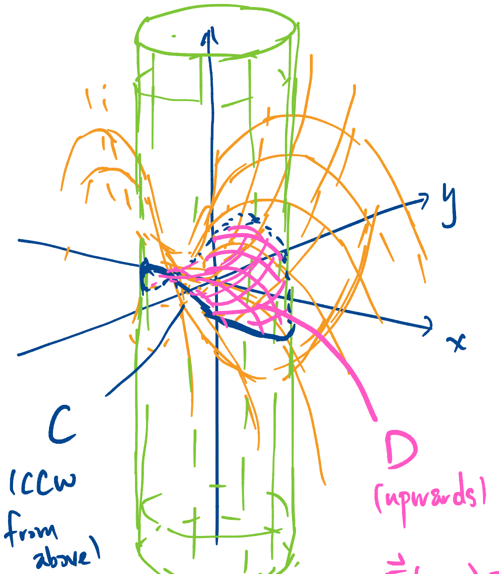
$$= \iint_{\text{some region in } u,v \text{ plane}} \left\langle \frac{y^2}{2}, 0, 1-x^2\right\rangle\cdot\langle -2,0,1\rangle\, du\,dv$$



$$= \iint_{1-u^2-v^2\geq 0} (1-u^2-v^2)\, du\, dv = \int_0^{2\pi}\int_0^1 (1-r^2)\, r\, dr\, d\theta$$

$$= \boxed{\frac{\pi}{2}}$$

Problem 2. Use Stokes' theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = \langle x^2y, \frac{1}{3}x^3, xy\rangle$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, ccw when viewed from above



$\vec{r}(x,y) = \langle x, y, y^2 - x^2\rangle$

$x^2+y^2 \le 1.$

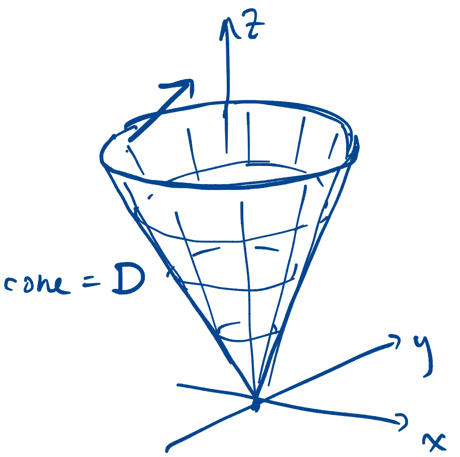
$\vec{r}(x,y) = \langle x, y, y^2-x^2\rangle$

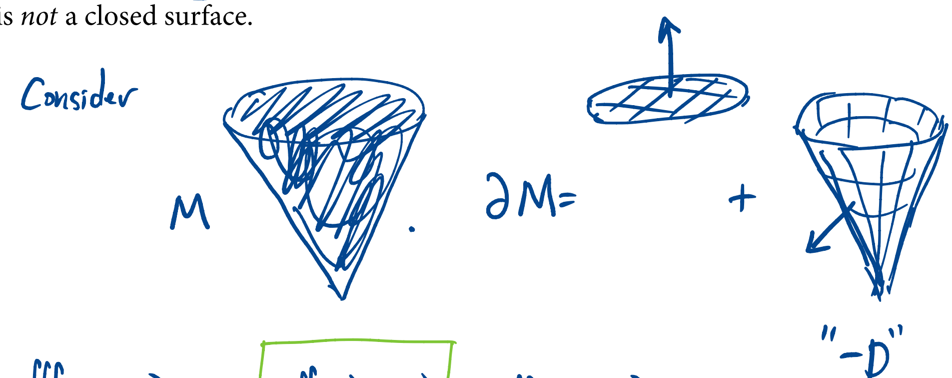
$$\int_C \vec{F}\cdot d\vec{r} = \iint_D (\nabla\times\vec{F})\cdot d\vec{S}$$

$$= \iint_{x^2+y^2\le 1} \underbrace{\langle x, -y, 0\rangle}_{\nabla\times\vec{F}} \cdot \underbrace{\langle 2x, -2y, 1\rangle}_{\vec{r}_x\times\vec{r}_y}\, dx\,dy$$

$$= \iint_{x^2+y^2\le 1} 2x^2 + 2y^2 \; dx\,dy = \int_0^{2\pi}\int_0^1 2r^2\, r\,dr\,d\theta = \boxed{\pi}$$

Problem 3. Consider the cone $z = \sqrt{x^2+y^2}$, $z \le 9$, oriented upwards. Use the divergence theorem to evaluate the flux of $\langle x, 0, 0\rangle$ through the cone. Note that the cone is *not* a closed surface.





$$\iiint_M \operatorname{div}\vec{F}\, dV = \boxed{\iint_{\substack{\text{lid}\\ \text{(upwards)}}} \vec{F}\cdot d\vec{S}} - \iint_D \vec{F}\cdot d\vec{S}$$

$\|$

0 because $\vec{F}$ parallel to lid.

Hence $\displaystyle\iint_D \vec{F}\cdot d\vec{S} = -\mathrm{Vol}(M)$

$$= -\frac{1}{3}\pi 9^2 \cdot 9 = \boxed{-243\pi}$$